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EMIL ERNST AND MICHEL VOLLE

Abstract. This article addresses a general criterion providing a zero duality gap for convex programs in the setting of the real locally convex spaces. The main theorem of our work is formulated only in terms of the constraints of the program, hence it holds true for any objective function fulfilling a very general qualification condition, implied for instance by standard qualification criteria of Moreau-Rockafellar or Attouch-Brézis type. This result generalizes recent theorems by Champion, Ban & Song and Jeyakumar & Li.

1. Introduction

Recently, several important breakthroughs (Champion [3] in 2004, Ban & Song [1] in 2009 and Jeyakumar & Li [8] in 2009), were made in characterizing convex programs with a zero duality gap. The aim of our work is to prove a very general result, implying all the above-mentioned theorems.

In order to clearly state the problems to which our work attempts to answer, we ask the reader to bear with us as we describe the main classical results in this field.

1.1. **Convex programs.** Throughout this article, X will be a locally convex space over the field of real numbers, and $(Y,\ \|\cdot\|,\ \leq_S)$ will be a real normed space endowed with the partial ordering associated to S, a closed convex and pointed cone of Y. To the space Y we add a singular element, denoted ∞_Y; we assume that the following conditions are satisfied:

$$y \leq_S \infty_Y,\ y + \infty_Y = \infty_Y\ \forall y \in Y, \quad r \cdot \infty_Y = \infty_Y\ \forall r \geq 0.$$

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Of special concern is the class of the S-convex mappings, that is functions $g : X \to Y \cup \{\infty_Y\}$, such that relation

$$g(\lambda x_1 + (1-\lambda)x_2) \leq_S \lambda g(x_1) + (1-\lambda) g(x_2) \tag{1}$$

holds true for any two vectors $x_1, x_2 \in \mathrm{dom}(g)$ and $\lambda \in [0,1]$, where $\mathrm{dom}(g) = \{x \in X : \ g(x) \neq \infty_Y\}$ is the effective domain of g. Another important class of mappings is composed from the S-lower semi-continuous (S-l.s.c. for short) functions; we recall that the function $g : X \to Y \cup \{\infty_Y\}$ is S-l.s.c. at $x_0 \in X$ if, for every $\varepsilon > 0$ and $y \in Y$, $y \leq_S g(x_0)$, there is $\mathcal{U}$, a neighborhood of x_0 , such that

$$g(\mathcal{U}) \subset ((y + \varepsilon \mathbb{B}_Y) + S) \cup \{\infty_Y\},$$

where $\mathbb{B}_Y$ is the closed unit ball from Y. It is easy to see that each and every of the sub-level sets $[g \leq_S y] = \{x \in X : g(x) \leq_S y\}$ of a S-convex and S-l.s.c. mapping is a closed and convex subset of X.

As customary, the set of all the mappings g defined over X with values within $Y \cup \{\infty_Y\}$, which are proper (that is $\mathrm{dom}(g) \neq \emptyset$) and S-convex is denoted by $\Lambda_0(X,Y,S)$; the class of all the functions from $\Lambda_0(X,Y,S)$ which are also S-l.s.c. is denoted by $\Gamma_0(X,Y,S)$. An important case is achieved when $(Y, \|\cdot\|, \leq_S) = (\mathbb{R}, |\cdot|, \leq)$; to simplify the notations, we write $\Lambda_0(X)$ and $\Gamma_0(X)$ instead of $\Lambda_0(X,\mathbb{R},\mathbb{R}_+)$ and $\Gamma_0(X,\mathbb{R},\mathbb{R}_+)$.

We are now in a position to define the main notion of our article.

Definition 1. *Let g be a mapping from $\Gamma_0(X,Y,S)$, and f be a function from $\Lambda_0(X)$ such that $\mathrm{dom}(f) \cap [g \leq_S 0] \neq \emptyset$. The problem $P(f,g)$, asking to minorize the objective function f over the closed and convex set of constraints $A = [g \leq_S 0]$, is called a convex program:*

$$P(f,g): \qquad \text{Find} \quad \mathrm{Inf}_A\, f, \quad \text{where} \quad A = \{x \in X : g(x) \leq_S 0\} \qquad ;$$

its value is denoted by $\mathrm{Inf}\, P(f,g)$.

Remark 1. *When $Y = \mathbb{R}^k$, $S = \mathbb{R}^k_+$, and $g(x) = (g_1(x), \ldots, g_k(x))$, $g_i \in \Gamma_0(X)$, we recover the classical notion of an ordinary convex program:*

$$P(f,g_i): \qquad \text{Find} \quad \mathrm{Inf}\,\{f(x) : \ g_1(x) \leq 0, \ldots, g_k(x) \leq 0\}.$$

Remark 2. *Definition 1 states that for every convex program it holds that*

$$\mathrm{dom}(f) \cap [g \leq_S 0] \neq \emptyset;$$

accordingly,

$$\mathrm{Inf}\ P(f,g) < +\infty. \tag{2}$$

A classical way to study $P(f,g)$ is to address the dual convex problem

$$D(f,g): \quad \text{Find} \quad \mathrm{Sup}\left\{\mathrm{Inf}\{f(x) + \langle \lambda, g(x)\rangle_Y :\ x \in X\} :\ \lambda \in S^+\right\} \qquad ;$$

here Y^* is the topological dual of Y, $\langle \cdot,\cdot\rangle_Y : Y^* \times Y \to \mathbb{R}$ represents the bilinear form between Y and Y^*, S^+ is the positive dual cone of S,

$$S^+ = \{y^* \in Y^* :\ \langle y^*, s\rangle_Y \geq 0 \quad \forall s \in S\},$$

and we make the convention that

$$\langle y^*, \infty_Y\rangle = +\infty \quad \forall y^* \in S^+. \tag{3}$$

Obviously, the dual convex problem of an ordinary convex program takes the form:

$$D(f,g_i): \ \mathrm{Sup}\left\{\mathrm{Inf}\left\{f(x) + \sum_{i=1}^{k} \lambda_i\, g_i(x) :\ x \in X\right\} :\ \lambda_1, \ldots \lambda_k \geq 0\right\}.$$

Remark 3. *The convention stated in the general case by relation (3) entails for the case of the dual problem of an ordinary convex program the standard convention asking that*

$$0 \cdot (+\infty) = +\infty.$$

The ground of our interest in the apparently more complicated dual problem is that, from a numerical point of view, it is much easier to compute its solution, Sup $D(f,g)$ than the solution Inf $P(f,g)$ of the convex program. We are thus interested in characterizing convex programs for which the duality gap $\delta(f,g) = \mathrm{Inf}\ P(f,g) - \mathrm{Sup}\ D(f,g)$ amounts to zero, simply because such programs are easier to solve.

A standard manner to address the duality gap of a convex program, is to study the extended-real-valued mapping $v : Y \to \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$, customary called the infimal (or marginal) function, and defined by the following formula:

$$v(y) = \mathrm{Inf}\{f(x) :\ x \in [g \leq y]\} \quad \forall y \in Y.$$

At this point of our study, we need to recall a central notion in convex analysis. Given Z a locally convex space, Z^* its topological dual and $\langle \cdot, \cdot \rangle_Z$ the bilinear form between Z and Z^*, to any extended-real-valued mapping $h : Z \to \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ we associate its Fenchel-Legendre conjugate, a function $h^* : Z^* \to \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ defined by the formula

$$h^*(z^*) = \text{Sup } \{\langle z^*, z\rangle_Z - h(z) : \ z \in Z\}.$$

Let us get back to the infimal function of the convex program, and consider $v^{**} : Y \to \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$, its Fenchel-Legendre bi-conjugate, as well as $\overline{v} : Y \to \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$, its lower semi-continuous envelope,

$$\overline{v}(y) = \liminf_{z \to y} \ v(z);$$

as v is convex, both v^{**} and $\overline{v}$ are convex functions which are lower semi-continuous, and it holds that

$$v^{**} \leq \overline{v} \leq v.$$

More precisely, it holds that either v^{**} does not take the value $-\infty$, and then $v^{**} = \overline{v}$, or $v^{**} = -\infty$, and then

$$\overline{v}(y) = \begin{cases} -\infty & y \in C \\ +\infty & y \notin C \end{cases}$$

for some closed and convex subset C of Y (which clearly depends upon the mappings f and g). It results that $v^{**}(y)$ and $\overline{v}(y)$ can differ ony when $v^{**}(y) = -\infty$ and $\overline{v}(y) = +\infty$, so every time when $v(y) < +\infty$, it follows that $v^{**}(y) = \overline{v}(y)$.

It can be proved (a very complete account of the problem may be found in [10, Chapters 2.6 and 2.9]), or in [2, Chapter 4.3] that, for each and every convex program $P(f, g)$, it holds that $v(0) = \text{Inf } P(f, g)$ and $v^{**}(0) = \text{Sup } D(f, g)$.

Moreover, relation (2) in Remark 2 reads that $v(0) < +\infty$; thus $v^{**}(0) = \overline{v}(0)$. We may hence conclude that ([2, Corollary 4.3.6]), the duality gap of a convex program is always non-negative, and amounts to zero if and only if the infimal function is lower semi-continuous at 0.

Remark 4. *If one allows convex programs for which the sets* $\text{dom}(f)$ *and* $[g \leq_S 0]$ *can be disjoint, then* $v(0)$ *may take the value* $+\infty$*, and so the duality gap may be non-null despite the lower semi-continuity at* 0 *of the infimal mapping.*

The characterization of convex programs with zero duality gap by using the lower semi-continuity of the infimal function is a profound, powerful and very elegant statement. It provides, in particular, easy and clear proofs of the fact that the duality gap is zero when the Slater's condition, $g(X) \cap \mathrm{Int}(S) \neq \emptyset$, is fulfilled, or when one of the mappings $f_\lambda(x) = f(x) + \langle \lambda, g(x) \rangle_Y$, $\lambda \in S^+$ is inf-compact (meaning that all its sub-level sets are compact subsets of X).

Yet, the marginal function depends upon both the objective function f, and the function g which expresses the constraints; hence, it is often difficult to decide whether v is l.s.c. or not at 0. An alternative to overcome this difficulty is to seek a zero duality gap criterion formulated only in terms of the mapping g. Such a criterion is bound to be less general, but, hopefully, it might be more amenable.

Following this line of reasoning, several authors have independently addressed the following problem over the last decade:

Find all the mappings g from $\Gamma_0(X, Y, S)$ such that the duality gap of the convex program $P(f, g)$ amounts to zero for each and every objective function f from $\Lambda_0(X)$, provided that f satisfies the qualification condition:

(Q1): f is finite and continuous at some point of A.

1.2. **Primal criterions: theorems by Champion and Ban & Song.** In his article [3], Champion tackles this problem in the particular case of ordinary convex programs, and achieves (Theorem 2.6) an answer, provided that X is a real normed space, and $f \in \Gamma_0(X)$.

In order to clearly state Champion's result, let us recall that the non-negative mapping $p \in \Gamma_0(X)$ penalizes in the sense of Motzkin ([9]) the objective function f from $\Lambda_0(X)$ if

$$\mathrm{Inf}_A\, f = \lim_{n\to\infty} \mathrm{Inf}_X(f + n\, p), \tag{4}$$

where $A = p^{-1}(0)$. As p is non-negative, it follows that

$$\mathrm{Sup}\,\{\mathrm{Inf}\,\{f(x) + s\, p(x) : \; x \in X\} \;:\; s \geq 0\} = \lim_{n\to\infty} \mathrm{Inf}_X(f + n\, p); \tag{5}$$

accordingly, p is a penalty function for f if and only if the convex program $P(f, p)$ (in this case $(Y, \|\cdot\|, \leq_S) = (\mathbb{R}, |\cdot|, \leq)$), has a zero duality gap: $\delta(f, p) = 0$.

Theorem [Champion, 2004]. *Let X be a real normed space, and consider a finite set $\{g_i : \ i = 1, \dots, k\}$ of functions from $\Gamma_0(X)$. The following sentences are equivalent.*

(i) *The duality gap of the ordinary convex program $P(f, g_i)$ is zero for every objective function f from $\Gamma_0(X)$ fulfilling condition $(Q1)$*

(ii) *The Courant-Beltrami mapping*

$$p_{CB}(x) = \max\{0, g_1(x), \dots, g_k(x)\} \tag{6}$$

penalizes every function f from $\Gamma_0(X)$ fulfilling condition $(Q1)$,

(iii) *Relation*

$$\overline{[p_{CB} \le 0] + L} = \bigcap_{\varepsilon>0} \overline{[p_{CB} \le \varepsilon] + L}$$

holds for every closed and linear subspace L of X.

The same assumptions (X is a real normed space, and $f \in \Gamma_0(X)$) underline the article of Ban & Song ([1]); their study extends Champion's result to the case of general convex programs, but only provided that the norm-interior of S is non-void, fact which allows us to define, for $y_1, y_2 \in Y$ the relation $y_1 < y_2$, as $y_2 - y_1 \in \mathrm{Int}(S)$.

Moreover, the authors provide no mapping of Courant-Beltrami type, fitted for the the general case.

Theorem [Ban & Song, 2009]. *Assume that X is a real normed space, and that the norm-interior of the cone S is non-void. The following sentences are equivalent.*

(i) *The duality gap of the convex program $P(f, g)$ is zero for every objective function f from $\Gamma_0(X)$ fulfilling condition $(Q1)$*

(ii) *Relation*

$$\overline{[g \le_S 0] + L} = \bigcap_{y>0} \overline{[g \le_S y] + L}$$

holds for every closed hyperplane L of X.

1.3. **A dual criterion: a theorem by Jeyakumar & Li.** A dual approach is given to this problem by Jeyakumar & Li ([8, Theorem 4.1]), at least for the case when X is a real Banach space and $f \in \Gamma_0(X)$. To this respect, the authors addressed the problem $P(-x^*, g)$, for each and every linear and continuous mapping x^* from X^*. One has clearly

$$\sigma_A(x^*) = -\mathrm{Inf}\, P(-x^*, g) \qquad \forall x^* \in X^*,$$

where σ_A is the support function of A:

$$\sigma_A : X^* \to \mathbb{R} \cup \{\infty\}, \quad \sigma_A(x^*) = \mathrm{Sup}_A \, x^*.$$

The authors denote by $h^\diamond(x^*)$ the opposite of the value of the dual problem whose objective function is $-x^*$: $h^\diamond(x^*) = -\mathrm{Sup}\, D(-x^*, g)$. Accordingly,

$$h^\diamond(x^*) = \mathrm{Inf}\left\{\mathrm{Sup}\{\langle x^*, x\rangle_X - \langle y^*, g(x)\rangle_Y : \ x \in X\} : \ y^* \in S^+\right\}$$

(as customary, X^* stands for the topological dual of X, and the bilinear form between X and X^* is denoted by $\langle \cdot, \cdot\rangle_X : X^* \times X \to \mathbb{R}$).

It is easy to see that, for any $g \in \Gamma_0(X, Y, S)$, $h^\diamond$ is a sublinear mapping; moreover, as the value of the primal problem always majorizes the value of the dual one, it results that

$$h^\diamond(x^*) \geq \sigma_A(x^*) \qquad \forall x^* \in X^*.$$

Theorem [Jeyakumar & Li, 2009]. *Assume that X is a real Banach space, and that the mapping g is such that $\langle \lambda, g(x)\rangle$ is a lower semi-continuous application for each and every vector λ from S^+. The following sentences are equivalent.*

(i) The duality gap of the convex program $P(f, g)$ is zero for every objective function f from $\Gamma_0(X)$ fulfilling condition $(Q1)$

(ii) The mapping $h^\diamond$ is lower semi-continuous.

1.4. **Plan and scope of the paper.** Like many works opening up a new field, the above-mentioned articles solve important problems, whereas rising pertinent questions.

a) Champion's theorem reposes upon the fact that the duality gap of an ordinary convex program $P(f, g_i)$ is zero if and only if the associates Courant-Beltrami mapping penalizes the objective function f.

Accordingly, Ban & Song's theorem may be viewed as an indication that a similar result may exists even for general convex programs (although the authors did not provided it), and clearly rises the following question:

Is it possible to define a mapping of Courant-Beltrami type for a convex program (non necessarily an ordinary one)? In other words, is it always possible to reduce the problem of the duality gap to a penalty problem?

b) Champions's and respectively Jeyakumar & Li's theorems provide a primal and respectively a dual scalar criteria for a zero duality gap convex program. It is thus a legitimate concern to ask whether:
Is there any relationship between the Courant-Beltrami mapping introduced by Champion (a mapping defined over the primal space X), and Jeyakumar & Li's application $h^\diamond$, defined over the dual space X^*?

c) The qualification condition $(Q1)$ is currently used in convex analysis, as it allows to achieve a large variety of results. However, many (if not all) among these results are valid provided that the underlining space X is locally convex, but not necessarily normed, and that the function f whose continuity at some point is required is convex, but not necessarily l.s.c.
Do the theorems by Champion, Ban & Song and Jeyakumar & Li remain true when X is a locally convex space, and the objective function f belongs only to $\Lambda_0(X)$? Is it possible to achieve similar theorems for other classical qualification condition, for instance of Attouch-Brézis type, or even in the absence of such a condition?

It is our aim to answer positively to these questions. To be more specific, Corollary 1 proves that a Courant-Beltrami map may be defined for any convex program, Theorem 2 shows that the mapping $h^\diamond$ is nothing but the directional derivative at 0 of the conjugate of the Courant-Beltrami map, and finally, Theorem 4 provides a family of very general zero duality gap criteria.

2. Courant-Beltrami mappings for general convex programs: an answer to questions a) and b)

2.1. **Definition and general properties of Courant-Berltrami functions.**

Definition 2. *Let g be a mapping from $\Gamma_0(X, Y, S)$. The function $p_{CB} : X \to \mathbb{R} \cup \{+\infty\}$ defined by the formula*

$$p_{CB}(x) = \begin{cases} \operatorname{dist}(g(x), -S) & x \in \operatorname{dom} g \\ +\infty & g(x) = \infty_Y \end{cases} \qquad \forall x \in X$$

is called the Courant-Beltrami function associated to g; as customary,

$$\operatorname{dist}(y, D) = \operatorname{Inf}\{\|y - z\| : \ z \in D\}$$

denotes the distance between a vector $y \in Y$ and a subset D of Y.

Remark 5. *If $Y=\mathbb{R}^n$, $S=\mathbb{R}^n_+$, and*

$$\|(y_1,\dots,y_n)\| = \max\{|y_i| :\ i=1\dots n\},$$

then the Courant-Beltrami mapping previously defined coincides with the one used by Champion.

The following result collects several classical properties of Courant-Beltrami mappings, whose standard proofs are omitted.

Proposition 1. *The Courant-Beltrami function associated to the mapping g from $\Gamma_0(X,Y,S)$ lies within the set $\Gamma(A)$, where*

$$\Gamma(A) = \left\{ f\in\Gamma_0(X) :\ f(x)\geq 0\ \forall x\in X,\ f^{-1}(0)=A \right\},$$

and $A=g^{-1}(-S)$.

An important feature of the newly defined map is provided by the following theorem.

Theorem 1. *The convex programs $P(f,g)$ and $P(f,p_{CB})$ share the same primal and dual values.*

Proof of Theorem 1: Let us denote by $v: Y\to\{-\infty\}\cup\mathbb{R}\cup\{\infty\}$ and respectively $v_{CB}:\mathbb{R}\to\{-\infty\}\cup\mathbb{R}\cup\{\infty\}$ the infimal functions associated to the convex programs $P(f,g)$ and respectively $P(f,p_{CB})$.

Set $y\in Y$; it is easy to prove that,

$$[g(x)\leq_S y] \Rightarrow [\mathrm{dist}(z,-S)\leq \|y\|];$$

accordingly,

$$v(y)\geq v_{CB}(\|y\|) \qquad \forall y\in Y. \tag{7}$$

Set now $a,\varepsilon>0$, and pick $x\in X$ such that

$$\mathrm{dist}(g(x),-S)\leq\ a, \quad f(x)\leq \left\{ \begin{array}{cc} v_{CB}(a)+\varepsilon & v_{CB}(a)>-\infty \\ -\frac{1}{\varepsilon} & v_{CB}(a)=-\infty \end{array}\right. ;$$

it easily yields that there is $y\in Y$ and $v\in -S$ such that

$$g(x)=y+v, \qquad \|y\|\leq a+\varepsilon.$$

As $g(x)\leq_S y$, it follows that $v(y)\leq f(x)$. We have thus proved that, for any $a,\varepsilon>0$,

$$\exists y\in(a+\varepsilon)\mathbb{B}_Y, \quad v(y)\leq \left\{ \begin{array}{cc} v_{CB}(a)+\varepsilon & v_{CB}(a)>-\infty \\ -\frac{1}{\varepsilon} & v_{CB}(a)=-\infty \end{array}\right. . \tag{8}$$

Relation (7) implies that $\overline{v}(0) \geq \overline{v_{CB}}(0)$, while from relation (8) we deduce that $\overline{v}(0) \leq \overline{v_{CB}}(0)$. As from Proposition 1 it follows that $v(0) = v_{CB}(0)$, it results that $\operatorname{Inf} P(f,g) = \operatorname{Inf} P(f,p_{CB})$ and $\operatorname{Sup} D(f,g) = \operatorname{Sup} D(f,p_{CB})$. □

The following obvious corollary of Theorem 1 establishes the relevance to our study of the above defined Courant-Beltrami map, and answers positively to question a).

Corollary 1. *Let $P(f,g)$ be a convex program. The two following statements are equivalent.*

(i) *The duality gap of $P(f,g)$ amounts to zero.*

(ii) *The Courant-Beltrami mapping associated to g penalizes f.*

2.2. **The directional derivate of the Courant-Beltrami mapping.** Let us consider a mapping $f : X \to \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$, x_0 a point at which f takes a real value, and v a vector from X; when it exists, the extended-real-valued limit

$$\lim_{r\to 0,\, r>0} \frac{f(x_0 + r\,v) - f(x_0)}{r}$$

is called the directional derivative of f along v. It is well-known that if f is a convex application, then the directional derivative exists along any vector from X; accordingly, at any point x_0 at which the convex function f takes a finite value, it is possible to consider the mapping $D_f(x_0) : X \to \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ (the directional derivative of f at x_0) defined by the following relation:

$$(D_f(x_0))(x) = \lim_{r\to 0,\, r>0} \frac{f(x_0 + r\,x) - f(x_0)}{r}.$$

The following two lemmas collect standard results concerning the directional derivative of the conjugate of an application from $\Lambda_0(X)$.

Lemma 1. *Let p be a function from $\Lambda_0(X)$ such that $\operatorname{Inf}_X p = 0$. Then*

$$(9) \qquad (Dp^*(0))\,(x^*) = -\lim_{n\to\infty} (\operatorname{Inf}_X(\langle -x^*, x\rangle_X + n\,p(x))).$$

Proof of Lemma 1: Since $\mathrm{Inf}_X\, p = 0$, it results that $p^*(0) = 0$. Accordingly,

$$
\begin{aligned}
(10)\ \mathrm{Inf}_X(\langle -x^*, x\rangle_X + n\,p(x)) &= -\mathrm{Sup}_X(\langle x^*, x\rangle_X - n\,p(x)) \\
&= -n\,\mathrm{Sup}_X\left(\left\langle \frac{x^*}{n}, x\right\rangle_X - p(x)\right) \\
&= -n\,p^*\left(\frac{x^*}{n}\right) \\
&= -\frac{p^*\left(\frac{x^*}{n}\right) - p^*(0)}{\frac{1}{n}}
\end{aligned}
$$

Relation (9) follows by letting n go to infinity in relation (10). □

Let A be a closed and convex non-void subset of X; its indicator function ι_A is currently defined as

$$\iota_A : X \to \mathbb{R} \cup \{+\infty\} \quad \iota_A(x) = \begin{cases} 0 & x \in A \\ +\infty & x \notin A \end{cases} ;$$

one has that $(\iota_A)^* = \sigma_A$ and $(\sigma_A)^* = \iota_A$. Remark also that ι_A belongs to the set $\Gamma(A)$.

Lemma 2. *Let p be a function from $\Gamma(A)$. Then*

$$(11) \qquad \overline{D_{p^*}(0)} = \sigma_A.$$

Proof of Lemma 2: Since $p^*(0) = -\inf_X p = 0$, one has that

$$(D_{p^*}(0))\,(x) = \inf_{n\in\mathbb{N}}\, n\,p^*\left(\frac{x}{n}\right),$$

and so

$$(D_{p^*}(0))^* = \sup_{n\in\mathbb{N}}\,(n\,p)^{**} = \sup_{n\in\mathbb{N}}\,(n\,p) = \iota_A.$$

Accordingly,

$$(12) \qquad (D_{p^*}(0))^{**} = \iota_A^* = \sigma_A.$$

We have thus proved that $D_{p^*}(0)$ is a convex mapping whose bi-conjugate does not achieve the value $-\infty$; hence

$$(13) \qquad \overline{D_{p^*}(0)} = (D_{p^*}(0))^{**}.$$

The conclusion of Lemma 2 stems by combining relations (12) and (13). □

The following result answers question b), by providing the researched connection between the penalty function of Courant-Beltrami type associated to a mapping g from $\Gamma_0(X, Y, S)$, and the mapping $h^\diamond$.

Theorem 2. *Let g be a mapping from $\Gamma_0(X,Y,S)$, and assume that the set $A = g^{-1}(-S)$ is non-void. Then*

$$D_{p^*_{CB}}(0) = h^\diamond. \tag{14}$$

Proof of Theorem 2: In view of the definition of the function $h^\diamond$, all what we have to prove is that

$$\left[D_{p^*_{CB}}(0)\right](x^*) = -\operatorname{Sup} D(-x^*, g) \qquad \forall x^* \in X^*. \tag{15}$$

As proved by Theorem 1, $\operatorname{Sup} D(-x^*, g) = \operatorname{Sup} D(-x^*, p_{CB})$. By applying relation (5) to the non-negative mapping p_{CB}, we deduce that

$$-\operatorname{Sup} D(-x^*, g) = -\lim_{n\to\infty} \operatorname{Inf}_X(\langle -x^*, x\rangle_X + n\, p_{CB}(x)), \tag{16}$$

and relation (15) easily follows from relation (16) and Lemma 1. $\square$

3. Towards an abstract zero duality gap criterion

In order to achieve a general zero duality gap criterion, let us first define and study a class of convex functions in relation with our main concern, namely the convex programs.

3.1. **A class of convex functions.** Given A a convex subset of X, let us denote by $\Xi(A)$ the class of all the functions f from $\Lambda_0(X)$ whose effective domain meets A and for which holds the following geometrical property:
(G): Let s be a real number such that $\operatorname{Inf}_X f < s < \operatorname{Inf}_A f$. Then, the set A may be strictly separated by the means of a closed hyperplane from the sub-level set $[f \le s]$ of f.

In other words, the convex mapping f is an element of the class $\Xi(A)$ if $\operatorname{Inf}_A f < +\infty$, and if, for any real number s inferior to $\operatorname{Inf}_A f$ and such that $[f \le s] \neq \emptyset$, the relation

$$\operatorname{Sup}\{\langle x^*, x\rangle_X : x \in [f \le s]\} < \operatorname{Inf}\{\langle x^*, x\rangle_X : x \in A\} \tag{17}$$

holds for some linear and continuous functional $x^* \in X^*$.

Remark 6. *It is easy to see that, when the set A is the singleton $\{x_0\}$ and $f \in \Lambda_0(X)$, then the two following statements are equivalent:*
(i): f belongs to $\Xi(A)$
(ii): f is l.s.c. at x_0.

Unlike for the case of a set A reduced to a point, no simple characterization of elements from $\Xi(A)$ seems to exists in general. This subsection uses standard convex analysis techniques to prove that several well-known sets of convex functions are contained in $\Xi(A)$.

To this end, let us first recall that a closed and convex subset C of a reflexive Banach space X is called slice-continuous if:

(a): its interior is non-void, and

(b): its support function σ_C is continuous at every element x^* of X^*, provided that $x^* \neq 0$.

In the case of the Euclidean spaces, the closed convex sets satisfying (b) were first studied in [7] under the name of continuous convex sets. The general Banach reflexive case was subsequently addressed in [5] and [6] (see also [4]).

The main interest of this class of convex sets is the following result ([5, Proposition 2, p. 194]): any two disjoint closed and convex nonempty subsets from a reflexive Banach space may be strictly separated by a closed hyperplane provided that at least one of them is slice-continuous.

It is now obvious that the following statement holds true.

Proposition 2. *Let X be a reflexive Banach space, and A a closed and convex subset of X. The following two sentences hold true.*

(i) Any mapping from $\Gamma_0(X)$ whose sub-level sets are slice-continuous and whose effective domain meets A belongs to $\Xi(A)$.

(ii) Any mapping from $\Gamma_0(X)$ whose effective domain intersects A lies within $\Xi(A)$, provided that A is slice-continuous.

In order to describe a second manner of tackling the problem of characterizing elements from $\Xi(A)$, let us recall that, given two mappings $f, g : X \to \mathbb{R} \cup \{+\infty\}$, it is customary to define the function $f \square g$, called the inf-convolution of f and g and defined as

$$(f \square g)(x) = \inf\{f(x+y) + g(-y) : \ y \in X\} \qquad \forall x \in X.$$

Lemma 3. *Any function f from $\Lambda_0(X)$ belongs to $\Xi(A)$ provided that its effective domain meets the set A and the following condition holds: (Q): $(f + \iota_A)^*(0) = (f^* \square \iota_A^*)(0)$.*

Proof of Lemma 3: Let f be a function from $\Lambda_0(X)$ whose effective domain meets A. Obviously, if $\operatorname{Inf}_A f = -\infty$, then f belongs to $\Xi(A)$.

Let us address the remaining case when $\operatorname{Inf}_A f \in \mathbb{R}$. Relation (Q) implies that, for any positive real number ε, there is an element x^*_ε of X^* such that

$$f^*(x^*_\varepsilon) + \iota^*_A(-x^*_\varepsilon) \leq \varepsilon - \operatorname{Inf}_A f. \tag{18}$$

Recall that

$$f^*(x^*_\varepsilon) = \operatorname{Sup}\{\langle x^*_\varepsilon, x\rangle - f(x) : \ x \in X\}$$

and that

$$\iota^*_A(-x^*_\varepsilon) = \operatorname{Sup}\{\langle x^*_\varepsilon; -y\rangle : \ y \in A\},$$

to infer from relation (18) that for every $x \in X$ and $y \in A$ it holds that

$$\langle x^*_\varepsilon, x - y\rangle - f(x) \leq \varepsilon - \operatorname{Inf}_A f. \tag{19}$$

Let now s be a real number such that $\operatorname{Inf}_X f < s < \operatorname{Inf}_A f$. Relation (19) implies that

$$\langle x^*_\varepsilon, x - y\rangle \leq s + \varepsilon - \operatorname{Inf}_A f$$

for each and every $x \in \ [f \leq s]$ and $y \in A$. It is now obvious that the linear and continuous mapping x^*_ε fulfills relation (17) provided that $0 < \varepsilon < \operatorname{Inf}_A f - s$. □

Apparently, using Lemma 3 with the intent to characterize functions from the class $\Xi(A)$ would lead us to replace the rather simple geometric condition (G) with the very technical statement (Q).

The utility of Lemma 3 become however more apparent in view the so-called theorems of qualification (a very complete account of this topic may be found in [10, Theorem 2.8.7]), which provide a large number of sufficient and esay to handle conditions for the validity of a relation much stronger than (Q).

Let us recall two among the most well-known such results.

Theorem. *Let X be a locally convex space, pick f, g two elements from $\Lambda_0(X)$, and assume that at least one of the two statements holds true.*

• *(Moreau-Rockafellar) f is finite and continuous at some point where g takes a real value*

•• *(Attouch-Brézis) X is a Banach space, $f, g \in \Gamma_0(X)$ and the cone $\mathbb{R}_+(\operatorname{dom}(f) - A)$ is a closed linear subspace of X.*

Then $(f + g)^* = f^* \Box g^*$, and the inf-convolution is exact, meaning that the infimum in the definition of $f^* \Box g^*$ is in fact a minimum.

Accordingly, by invoking Moreau-Rockafellar's theorem jointly with Lemma 3, one can establish the following result.

Proposition 3. *Let X be a locally convex space, and A a closed and convex subset of X. Any function f from $\Lambda_0(X)$ belongs to $\Xi(A)$ provided that the following qualification condition holds:*
(Q1): f is finite and continuous at some point of A.

The next statement is an obvious consequence of Proposition 3. It can also be easily checked directly.

Proposition 4. *The class $\Xi(A)$ contains each and every linear and continuous functional x^* from X^*.*

When the underline space X is a real Banach space, and $f \in \Gamma_0(X)$, then property **(G)** is implied by a more general condition, of Brezis-Attouch type.

Proposition 5. *Let $(X, \|\cdot\|)$ be a real Banach space, and A be one of its closed and convex subsets. Any function f from $\Gamma_0(X)$ belongs to $\Xi(A)$ provided that the following qualification condition holds:*
(Q2): $\mathbb{R}_+(\mathrm{dom}(f) - A)$ is a non-void closed linear subspace of X.

3.2. **An abstract zero duality gap theorem.** Our main result provides an abstract version of the above-mentioned results by Champion, Ban & Song and Jeyakumar & Li.

Theorem 3. *Let X be a locally convex space, A a closed and convex subset of X, p a mapping from $\Gamma(A)$ and $\mathcal{F}$ a subset of $\Xi(A)$ which contains X^*. The following four statements are equivalent.*

(i): The function p is a penalty functional for all the mappings $f \in \mathcal{F}$

(ii) The directional derivative at 0 of p^*, the conjugate of p, coincides with σ_A, the support functional of A

(iii): The directional derivative at 0 of p^* belongs to $\Gamma_0(X^*)$

(iv): For any L, closed linear subspace of X it holds that

$$\overline{[p \leq 0] + L} = \bigcap_{\varepsilon > 0} \overline{[p \leq \varepsilon] + L}. \tag{20}$$

Proof of Theorem 3: We shall prove that each and every of the statements (i), (iii) and (iv) is equivalent with the statement (ii).

$(i) \Rightarrow (ii)$ Let $x^* \in X^*$; p is a penalty function for all the mappings from $\mathcal{F}$, and in particular for $-x^*$. It results that

$$\mathrm{Inf}_A \langle -x^*, x\rangle = \lim_{n\to\infty} \left(\mathrm{Inf}_X(\langle -x^*, x\rangle + n\, p(x))\right). \tag{21}$$

But

$$\mathrm{Inf}_A \langle -x^*, x\rangle = -\mathrm{Sup}_A \langle x^*, x\rangle = -\sigma_A(x^*), \tag{22}$$

while Lemma 1 proves that

$$\left(Dp^*(0)\right)(x^*) = -\lim_{n\to\infty} \left(\mathrm{Inf}_X(\langle -x^*, x\rangle + n\, p(x))\right). \tag{23}$$

Finally, from relations (21), (22) and (23) it follows that

$$\sigma_A(x^*) = \left(Dp^*(0)\right)(x^*) \qquad \forall x^* \in X^*.$$

$(ii) \Rightarrow (i)$ Suppose, to the end of obtaining a contradiction, that there is a function f belonging to the class $\Xi(A)$ and two real numbers s_1 and s such that $s_1 < s < \mathrm{Inf}_A f$ and

$$\mathrm{Inf}_X(f(x) + n\, p(x)) \leq s_1 \qquad \forall n \in \mathbb{N}. \tag{24}$$

Accordingly, there is a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that

$$f(x_n) + n\, p(x_n) < s \qquad \forall n \in \mathbb{N}. \tag{25}$$

As p takes only positive values, it results that $(x_n)_{n\in\mathbb{N}} \subset [f \leq s]$.

The sub-level set $[f \leq s]$ is thus non-void, $s < \mathrm{Inf}_A f$, and $f \in \Xi(A)$. It follows that the set $[f \leq s]$ may be strictly separated by the means of a closed hyperplane from the set A: there are two real numbers a and b and a linear and continuous functional $x^* \in X^*$ such that

$$\mathrm{Sup}\{\langle x^*,\, x\rangle :\ x \in [f \leq s]\} < a < b < \mathrm{Inf}\{\langle x^*,\, x\rangle :\ x \in A\}. \tag{26}$$

Our claim is that $\sigma_A(-x^*) < (D_{p^*}(0)(-x^*)$.

Indeed, from relation (26) it stems that

$$b < -\sigma_A(-x^*). \tag{27}$$

In order to estimate $D_{p^*}(0)(-x^*)$, let us pick $x_0 \in \mathrm{dom}\, f \cap A$, and define the real numbers

$$\lambda_n = \frac{\langle x^*, x_0\rangle - a}{\langle x^*, x_n - x_0\rangle} \qquad \forall n \in \mathbb{N}.$$

When combined with the fact that $x_0 \in A$, relation (26) implies that $\langle x^*, x_0 \rangle > b$; when we take into account that $(x_n)_{n\in\mathbb{N}} \subset [f \le s]$, the same relation yields that $\langle x^*, x_n \rangle < a$ for all $n \in \mathbb{N}$. Clearly, this means that all the real numbers λ_n belong to $[0, 1]$, and that

$$\langle x^*, y_n \rangle = a \qquad \forall n \in \mathbb{N}, \tag{28}$$

where $y_n = \lambda_n \, x_n + (1 - \lambda_n) \, x_0$.

Combine once again relation (26), this time with relation (28), to deduce that

$$f(y_n) > s \qquad \forall n \in \mathbb{N}, \tag{29}$$

and use Jensen's inequality for the convex functions f and respectively p, to infer that

$$f(y_n) \le \lambda_n \, f(x_n) + (1 - \lambda_n) \, f(x_0) \qquad \forall n \in \mathbb{N}, \tag{30}$$

and respectively that

$$p(y_n) \le \lambda_n \, p(x_n) \qquad \forall n \in \mathbb{N} \tag{31}$$

(recall that $p(x_0) = 0$, as $x_0 \in A = p^{-1}(0)$).

From relations (25), (29) and (30) it yields that

$$0 < (1 - \lambda_n)(f(x_0) - s) - n \, \lambda_n \, p(x_n) \qquad \forall n \in \mathbb{N}$$

inequality which, combined with relation (31) proves that

$$0 \le p(y_n) < \frac{(1 - \lambda_n)(f(x_0) - s)}{n} \qquad \forall n \in \mathbb{N}.$$

But for any $n \in \mathbb{N}$ we have that $0 \le \lambda_n \le 1$; thus

$$0 \le \frac{(1 - \lambda_n)(f(x_0) - s)}{n} \le \frac{f(x_0) - s}{n}.$$

Accordingly, $\lim_{n\to+\infty} p(y_n) = 0$. Pick $m \in \mathbb{N}$; then

$$\lim_{n\to\infty} (\langle x^*, y_n \rangle + m \, p(y_n)) = a \qquad \forall m \in \mathbb{N}. \tag{32}$$

As a consequence of relation (32) it results that

$$\operatorname{Inf}_X (\langle x^*, \, x \rangle + n \, p(x)) \le a \qquad \forall n \in \mathbb{N}, \tag{33}$$

and Lemma 1 together with relation (33) reads that

$$-(D_{p^*}(0)(-x^*) \le a. \tag{34}$$

From relations (34) and (27) it results that $\sigma_A(-x^*) < (D_{p^*}(0)(-x^*)$, fact which contradicts assumption (ii). Statement (24) is accordingly false, so relation (i) is true.

$(ii) \Rightarrow (iii)$ This implication is obvious.

$(iii) \Rightarrow (ii)$ This implication is an easy consequence of Lemma 2.

$(ii) \Rightarrow (iv)$ Let L, a closed linear subspace of X, and assume, to the purpose of achieving a contradiction, that there is some point $x_0 \in X$ such that

$$x_0 \in \bigcap_{\varepsilon>0} \overline{[p \leq \varepsilon] + L}, \qquad x_0 \notin \overline{A+L}. \tag{35}$$

In the framework of the real locally convex spaces, it is always possible to strictly separate by the means of a closed hyperplane a point from a disjoint closed and convex set. In particular, this statement is valid for the point x_0 and for the closed and convex set $\overline{A+L}$; hence

$$\langle x^*, x_0\rangle < \operatorname{Inf}\left\{\langle x^*, x\rangle : \ x \in \overline{A+L}\right\} \tag{36}$$

for some linear and continuous functional $x^* \in X^*$. Let us also remark that relation (36) implies that

$$\langle x^*, x\rangle = 0 \qquad \forall x \in L. \tag{37}$$

We claim that $(D_{p^*}(0))(-x^*) < \sigma_A(-x^*)$.

Indeed, as $A \subset \overline{A+L}$, from relation (36) it yields that

$$\langle x^*, x_0\rangle < -\sigma_A(-x^*). \tag{38}$$

In order to estimate $(D_{p^*}(0))(-x^*)$, let us pick $\eta > 0$ and $n \in \mathbb{N}$, and apply relation (35) for $\varepsilon = \frac{\eta}{n}$. It results that there are two nets $(y_i)_{i\in I} \subset X$ and $(z_i)_{i\in I} \subset L$, such that

$$x_0 = \lim_{i\in I}(y_i + z_i), \quad p(y_i) \leq \frac{\eta}{n} \ \forall i \in I. \tag{39}$$

In view of relations (37) and (39), we deduce that

$$\lim_{i\in I} \langle x^*, y_i\rangle = \lim_{i\in I} \langle x^*, y_i + z_i\rangle = \langle x^*, x_0\rangle\, ; \tag{40}$$

as $n\, p(y_i) \leq \eta$, it follows that

$$\operatorname{Inf}_X(\langle w^*, x\rangle + n\, p(x)) \leq \langle x^*, x_0\rangle + \eta \qquad \forall n \in \mathbb{N},\ \forall \eta > 0. \tag{41}$$

Let η go to 0 in relation (41); then

$$\operatorname{Inf}_X(\langle x^*, x\rangle + n\, p(x)) \leq \langle x^*, x_0\rangle \qquad \forall n \in \mathbb{N}. \tag{42}$$

From relation (42) and Lemma 1 it results that

$$-(D_{p^*}(0))(-x^*) \leq \langle x^*, x_0 \rangle \,. \tag{43}$$

Our claim follows from relations (38) and (43); this fact contradicts assumption (*ii*). We may henceforth state that relation (35) is false; accordingly, the statement (*iv*) holds true.

(*iv*) $\Rightarrow$ (*ii*) To the purpose of achieving a contradiction, let us assume that there exists a linear and continuous functional $x^* \in X^*$ such that

$$(D_{p^*}(0))(x^*) > \sigma_A(x^*). \tag{44}$$

Set $L = \{x \in X : \ \langle x^*, x \rangle = 0\}$, and pick $x_0 \in X$ such that

$$(D_{p^*}(0))(x^*) > \langle x^*, x_0 \rangle > \sigma_A(x^*). \tag{45}$$

It is obvious that

$$\langle x^*, x \rangle \leq \sigma_A(x^*) \qquad \forall x \in \overline{A+L}; \tag{46}$$

relation (46), combined with the second inequality of relation (45) implies that

$$x_0 \notin \overline{A+L}. \tag{47}$$

We claim that

$$x_0 \in [p \leq \varepsilon] + L \qquad \forall \varepsilon > 0; \tag{48}$$

in other words, we want to prove that, for any $\varepsilon > 0$, there is $x_\varepsilon \in X$ such that

$$\langle x^*, x_0 \rangle = \langle x^*, x_\varepsilon \rangle \,, \qquad p(x_\varepsilon) \leq \varepsilon. \tag{49}$$

To the end of achieving a contradiction to this new claim, let us assume that there is a real positive number a such that

$$p(x) \geq a \qquad \forall x \text{ s.t. } \langle x^*, x \rangle = \langle x^*, x_0 \rangle \,. \tag{50}$$

Pick $x \in X$ such that $\langle x^*, x \rangle \geq \langle x^*, x_0 \rangle$; our aim is to estimate $p(x)$. To this end, let us consider an element $x_1 \in A$; from relation (45) it follows that

$$p(x_1) = 0, \qquad \langle x^*, x_1 \rangle < \langle x^*, x_0 \rangle \,. \tag{51}$$

The inequality in relation (51) allows us to define the element

$$y = \frac{\langle x^*, x - x_0 \rangle}{\langle x^*, x - x_1 \rangle}\, x_1 + \frac{\langle x^*, x_0 - x_1 \rangle}{\langle x^*, x - x_1 \rangle}\, x,$$

and to remark that $\langle x^*, y\rangle = \langle x^*, x_0\rangle$. Relation (50) and the equality in relation (51) may now be used to show that

$$\begin{aligned} a \le p(y) &\le \frac{\langle x^*, x - x_0\rangle}{\langle x^*, x - x_1\rangle}\, p(x_1) + \frac{\langle x^*, x_0 - x_1\rangle}{\langle x^*, x - x_1\rangle}\, p(x) \\ &= \frac{\langle x^*, x_0 - x_1\rangle}{\langle x^*, x - x_1\rangle}\, p(x). \end{aligned} \tag{52}$$

Accordingly,

$$p(x) \ge a\left(1 + \frac{\langle x^*, x - x_0\rangle}{\langle x^*, x_0 - x_1\rangle}\right) \qquad \forall x \text{ s.t. } \langle x^*, x\rangle \ge \langle x^*, x_0\rangle . \tag{53}$$

An easy computation proves that, for any $n \in \mathbb{N}$ and $x \in X$ such that $\langle x^*, x\rangle \ge \langle x^*, x_0\rangle$, it holds that

$$\begin{aligned} \langle -x^*, x\rangle + n\,p(x) &\ge na - \langle x^*, x_0\rangle \\ &+ \langle x^*, x - x_0\rangle \left(\frac{na}{\langle x^*, x_0 - x_1\rangle} - 1\right). \end{aligned} \tag{54}$$

Set n_0 an integer such that $n_0 \ge \frac{\langle x^*, x_0 - x_1\rangle}{a}$; relation (54) proves that

$$\langle -x^*, x\rangle + n\,p(x) \ge -\langle x^*, x_0\rangle \tag{55}$$

for any $n \ge n_0$ and $x \in X$ such that $\langle x^*, x\rangle \ge \langle x^*, x_0\rangle$. As, for any integer n and any $x \in X$ such that $\langle x^*, x\rangle < \langle x^*, x_0\rangle$, it is obvious that

$$\langle -x^*, x\rangle + n\,p(x) \ge -\langle x^*, x_0\rangle , \tag{56}$$

one obtains, by combining relations (55) and (56), that

$$\langle -x^*, x\rangle + n\,p(x) \ge -\langle x^*, x_0\rangle \qquad \forall n \ge n_0\ \forall x \in X. \tag{57}$$

Lemma 1 together with relation (57) yield that

$$(D_{p^*}(0))(x^*) \le \langle x^*, x_0\rangle ; \tag{58}$$

the contradiction between the first inequality in relation (45) and relation (58) proves that relation (50) is false. In other words, our claim (48) holds true, fact which means that

$$x_0 \in \bigcap_{\varepsilon > 0} ([p \le \varepsilon] + L). \tag{59}$$

Relations (47), (59) and assumption (iv) are mutually incompatible; this second contradiction proves that relation (44) is false, so statement (ii) holds true. $\square$

4. A general zero duality gap theorem: an answer to question c)

The following result, whose proof is a direct consequence of the results in the previous sections, generalizes the theorems of Champion, Ban & Song and Jeyakumar & Li, answering positively the third question of our study.

Theorem 4. *Let $g \in \Gamma_0(X, Y, S)$ such that the set $A = g^{-1}(-S)$ is non-void, and let $\mathcal{F}$ be one of the following sets of functions:*

α) given X a locally convex space, $\mathcal{F}$ is the set of all the elements from $\Lambda_0(X)$ which are finite and continuous at some point of A

β) given a Banach space X, $\mathcal{F}$ is the set of all the elements f from $\Gamma_0(X)$ such that $\mathbb{R}_+(\mathrm{dom}(f) - A)$ is a non-void closed subspace of X.

The following four statements are equivalent.

(i): The duality gap of the convex program $P(f, g)$ is zero for any objective function f belonging to $\mathcal{F}$

(ii) The Courant-Beltrami functional p_{CB} penalizes all the mappings f from $\mathcal{F}$

(iii): For any L, closed linear subspace of X it holds that

$$\overline{[p_{CB} \leq 0] + L} = \bigcap_{\varepsilon > 0} \overline{[p_{CB} \leq \varepsilon] + L}.$$

(iv) The directional derivative at 0 of p_{CB}^*, the conjugate of p_{CB}, is a lower semi-continuous application.

Moreover, when, given a reflexive Banach space X, $\mathcal{F}$ is the set of all the elements from $\Gamma_0(X)$ whose effective domain meets A and whose sub-level sets are slice-continuous, then the equivalent statements (iii) and (iv) imply both statements (i) and (ii).

References

[1] L. Ban, W. Song, *Duality gap of the conic convex constrained optimization problems in normed spaces*, Math. Program. **119** (2009), p. 195-214.

[2] J. Borwein, A. Lewis, *Convex Analysis and Nonlinear Optimization: Theory and Examples*, Birkhäuser, 2006.

[3] T. Champion, *Duality gap in convex programming*, Math. Program. **99** (2004), p. 487-498.

[4] P. Coutat, M. Volle, J.-E. Martínez-Legaz, *Convex functions with continuous epigraph or continuous level sets*, J. Optim. Theory Appl. **88**(1996), p. 365-379.

[5] E. Ernst, M. Théra, C. Zălinescu, *Slice-continuous sets in reflexive Banach spaces: convex constrained optimization and strict convex separation*, J. Funct. Anal. **223** (2005), p. 179-203.

[6] E. Ernst, M. Théra, *Slice-continuous sets in reflexive Banach spaces: Some complements* Set-Valued Anal. **16** (2008), p. 307-318.

[7] V. Gale, V. Klee, *Continuous convex sets*, Math. Scand. **7** (1959), p. 379-391.

[8] V. Jeyakumar, G.Y. Li, *New dual constraint qualifications characterizing zero duality gaps of convex programs and semidefinite programs*, Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods **71** (2009), p. 2239-2249.

[9] T.S. Motzkin, *New technique for linear inequalities and optimisation*, Project SCOOP Symposium on Linear Inequalities and Programming, Planning Research Division, U.S. Air Force, Washington D.C., 1952.

[10] C. Zălinescu, *Convex analysis in general vector spaces*, World Scientific, Singapore, 2002.

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